ROBUST MULTIPLE OBJECTIVE CONTROL BY USING LMI OPTIMIZATION

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Abstract

The present paper considers the problem of designing guaranteed cost controllers minimizing multiple quadratic performance objectives, for linear systems with parametric uncertainties. The design procedure uses the LMI optimization technique. By setting the multi-objective control problem in an LMI form, one may find the suitable uncertainty decomposition which allows to compute a positive definite solution satisfying multiple modified Riccati equations. In that sense, a single quadratic regulator is used to optimize multiple performance indices. Furthermore, it ensures a minimal guaranteed cost bound over all the performance indices. The proposed approach is illustrated by a numerical example with two quadratic performance objectives.

1 Introduction

Robust controller design of linear systems with parametric uncertainty in the LQR framework leads to the so-called *guaranteed cost control* (GCC), introduced by Chang and Peng [3] and since then readdressed by several authors (see [4] for details and related references). The GGC approach consists in designing a linear, constant gain state-feedback controller which guarantees the closed-loop system's stability and a certain level of performance for all system model uncertainty of a given class. This is achieved by minimizing an upper bound of the quadratic performance objective associated with the optimization problem. Moreover, as shown in [8], the nominal system's optimality is preserved despite uncertainty in the sense of the well known stability margins ensured by the optimal linear quadratic regulator [1]. Some questions related with the existence and computation of the solution of the modified Riccati equation which occurs in the guaranteed cost control problem have been studied in [7] and furthermore in [6], in the LMI optimization framework. However, in many practical design problems one has to cope not only with uncertainty but also with multiple performance objectives. This problem has been addressed by Dorato *et al.* [5]; Pareto optimal solutions have been proposed by using linear bounds of uncertainty.

In the present paper the robust multi-objective GCC problem is studied in the LMI framework. Non-linear bounds of uncertainty are used, obtained from a rank-1 decomposition of the uncertainty matrix. Since that is a non-unique decomposition, it offers several degrees of freedom in the design procedure. By solving an LMI objective minimization problem [2], one may find the "optimal" rank-1 uncertainty decomposition which leads to the control law guaranteeing a minimal upper bound over all performance objectives.

2 Problem formulation

Consider the linear uncertain system in state-space representation

$$\dot{x}(t) = (A + \Delta A(t))x(t) + (B + \Delta B(t))u(t) \qquad (1)$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control vector, A and B are the state and control matrices, respectively, having appropriate dimensions.

The system uncertainty is described by

$$\Delta A(t) = \sum_{i=1}^{p} \alpha_i(t) A_i \qquad \alpha_i^2(t) \leq 1$$
$$\Delta B(t) = \sum_{i=1}^{q} \beta_i(t) B_i \qquad \beta_i^2(t) \leq 1 \qquad (2)$$

where the scalars $\alpha_i(t)$, $\beta_i(t)$ are uncertain parameters, possibly time-varying, belonging to specified ranges, and A_i, B_i are given constant matrices determining the uncertainty structure. Without loss of generality, one can allways assume that A_i, B_i have unity rank and thus they may be decomposed in form of products of vectors of appropriate dimensions, as follows:

$$A_i = d_i e_i^T \quad ; \quad B_i = f_i g_i^T \qquad (3)$$

This is called the *rank-1 decomposition*. By using these vectors, define the matrices

$$\begin{array}{ll} D := [d_1 ... d_p] & F := [f_1 ... f_q] \\ E := [e_1 ... e_p]^T & G := [g_1 ... g_q]^T \\ S := diag(s_1 ... s_p) & T := diag(t_1 ... t_q) \end{array} \qquad (4)$$

where s_i, $i = 1, ..., p$ and t_i, $i = 1, ..., q$ are positive scalars. Since the decomposition (3) is not unique, these scalars may be chosen to determine a suitable rank-1 decomposition in order to satisfy different design requirements. In other words the elements of S, T will be treated as free parameters, in the design procedure.

Consider also multiple quadratic performance indices of the form

$$J_i := \int_0^\infty [x^T(t) Q_i x(t) + u^T(t) R_i u(t)] dt \qquad (5)$$

with $Q_i > 0$, $R_i > 0$, $i = 1, ..., r$.

The *multiple objective guaranteed cost control* problem consists in finding a linear state-feedback control law of the form

$$u(t) = -Kx(t) \qquad (6)$$

such that the corresponding values of **all** performance indices are made as small as possible and remain upper bounded for all parameter uncertainties consistent with (2).

3 LMI optimization approach

3.1 The single-objective case

Let a single performance index

$$J := \int_0^\infty [x^T(t) Q x(t) + u^T(t) R u(t)] dt \qquad (7)$$

have to be minimized. Then, the problem reduces to the *standard guaranteed cost control*. As shown in [7] and [6] the *guaranteed cost control* law of the form

$$u^*(t) = -\delta R^{-1} B^T P x(t) \qquad (8)$$

ensures an upper bound, called *guaranteed cost*, of the form

$$J(u^*(t)) \leq J^* = x(0)^T P x(0) \qquad (9)$$

of the quadratic performance index (7) for all parameter variations consistent with (2). The (nxn) matrix P is the positive definite solution of the modified Riccati equation associated with the GCC problem

$$\begin{array}{l} A^T P + PA - P[\delta(BR^{-1}B^T - \\ \delta BR^{-1}G^T T^{-1} G R^{-1} B^T) \\ -FTF^T - DSD^T]P + E^T S^{-1} E + Q = 0 \end{array} \qquad (10)$$

The positive scalar δ and the scaling matrices S, T are chosen by the designer. For random initial state with $E[x(0)x^T(0)] = I$, where E denotes the expected value and I denotes the identity matrix, one considers the expected value of the prformance index, i.e.

$$J = E\{\int_0^\infty [x^T(t) Q x(t) + u^T(t) R u(t)] dt\} \qquad (11)$$

Then the guaranteed cost will be

$$J^* = tr(P) \qquad (12)$$

where $tr(.)$ denotes the trace. Since the GCC problem is often related with conservatism, i.e. the resulting upper bounds are too large with respect to the minimal J obtained from the LQR optimal design for the system without uncertainty, it is desirable to make J^* as small as possible. This leads to an auxilliary optimization problem which is often analytically not tractable. An efficient solution has been proposed in [6] by solving an LMI objective minimization problem.

3.2 The multi-objective case

Consider now the multiple quadratic performance indices of (5). For random initial conditions one obtains

$$J_i := E\{\int_0^\infty [x^T(t) Q_i x(t) + u^T(T) R_i u(t)] dt\} \qquad (13)$$

for $i = 1, ..., r$. In order to find a control law (6) ensuring an upper bound for all the above indices one has to find a positive definite matrix P and positive scalars δ_i, if they exist, such that the following set of modified Riccati equations are satisfied

$$\begin{array}{l} A^T P + PA - P[\delta_i(BR_i^{-1}B^T - \\ \delta_i BR_i^{-1} G^T T_i^{-1} G R_i^{-1} B^T) \\ -FT_iF^T - DS_iD^T]P + E^T S_i^{-1} E + Q_i = 0 \end{array} \qquad (14)$$

for $i = 1, ..., r$. Then, the resulting guaranteed cost control law will be of the form

$$u^*(t) = -\delta_i R_i^{-1} B^T P x(t) \qquad (15)$$

ensuring that

$$J_i \leq J^* = tr(P) \qquad (16)$$

for all $i = 1, ..., r$.

Remark 3.3

Conditions for the existence of a closed form positive definite solution P of (14) have not been given, by now. In consequence, it has to be solved by *trial and error* for different choices of the scalars δ_i. Moreover, different possible uncertainty decompositions can be used by means of the choice of S_i, T_i providing additional degrees of freedom to the solution procedure. Hence, the computation of P becomes very complex. In the sequel, a numerical solution is proposed in terms of LMI optimization.

Theorem 3.4

Consider the uncertain system (1)-(4) and multiple performance indices (13). A guaranteed cost control law (15) minimizing the upper bound (16), for all uncertainties consistent with (2), exists, if the LMI minimization problem

$$\min_{M,W,S_i,T_i,\delta_i} tr(M) \tag{17}$$

$$\begin{pmatrix} M & I \\ I & W \end{pmatrix} > 0 \tag{18}$$

$$\begin{pmatrix} \mathcal{L}_1(W) & WE^T & \delta_1 BR_1^{-1}G^T & W \\ EW & S_1 & 0 & 0 \\ GR_1^{-1}B^T\delta_1 & 0 & T_1 & 0 \\ W & 0 & 0 & \bar{Q}_1^{-1} \end{pmatrix} > 0 \tag{19}$$

$$\vdots$$

$$\begin{pmatrix} \mathcal{L}_r(W) & WE^T & \delta_r BR_r^{-1}G^T & W \\ EW & S_r & 0 & 0 \\ GR_r^{-1}B^T\delta_r & 0 & T_r & 0 \\ W & 0 & 0 & \bar{Q}_r^{-1} \end{pmatrix} > 0 \tag{20}$$

with

$$\mathcal{L}_1(W) = -WA^T - AW + \delta_1 BR_1^{-1}B^T - DS_1D^T - FT_1F^T \tag{21}$$

and

$$\mathcal{L}_r(W) = -WA^T - AW + \delta_r BR_r^{-1}B^T - DS_rD^T - FT_rF^T \tag{22}$$

has a non-empty set of feasible solutions $(M, W, S_i, T_i, \delta_i)$, $i = 1, ..., r$, where M, W are symmetric positive definite matrices, S_i, T_i are diagonal positive definite matrices, δ_i are positive scalars and $\bar{Q}_i < Q_i$, $i = 1, ..., r$. Then, the positive definite matrix P in (15), (16) is

$$P = W^{-1} \tag{23}$$

Proof

By applying the Shur complement [2] to the multiple LMIs (19)...(20) and then pre- and post- multiplying by P, one obtains the following set of inequalities:

$$A^TP + PA - P[\delta_i(BR_i^{-1}B^T - \delta_i BR_i^{-1}G^TT_i^{-1}GR_i^{-1}B^T) - FT_iF^T - DS_iD^T]P + E^TS_i^{-1}E + \bar{Q}_i < 0 \tag{24}$$

for $i = 1, ..., r$.
The set of equations (14) are then satisfied for $Q_i > \bar{Q}_i$. The values of S_i, T_i are specified from the LMI optimization procedure. Application of the Schur complement to the LMI (18) yields

$$M > W^{-1} = P \tag{25}$$

Consequently, minimizing the trace of M implies minimization of the trace of P. Since P is the common solution of the set of LMIs (20), $tr(P)$ is the minimal *guaranteed cost bound* of the multiple performance objectives.
□

4 Numerical example

The above result is illustrated by an example. The same uncertain system is also studied in [5]. Consider a 3rd order system of the form of (1) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.6 \end{pmatrix}$$

Consider also two quadratic performance indices with weighting matrices

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix},\ Q_2 = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

$R_1 = R_2 = 1$

Let the uncertainty decomposition matrices be as follows:

$$D = \begin{pmatrix} 0 \\ 0 \\ 1.6 \end{pmatrix},\ E = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},\ F = G = 0$$

Application of the proposed approach to this example results to $tr(P) = 17.13$ for $\delta = 1$ and $S_1 = S_2 = 0.185$. Note that the the optimal quadratic performance values for the system without uncertainty i.e. by solving the LQR problem are, respectively, $J_{1min} = 7.29$ and $J_{2min} = 4.37$. On the other hand, by solving separately two GCC problems one finds the two GC bounds as $J_1^* = 16.00$ and $J_2^* = 10.12$. Obviously, the common GC bound $tr(P)$ is found to be slightly larger but not conservative.
The bound obtained by using the Pareto optimization approach [5] is found to vary between 65 and 75, depending on the values of specific design parameters. It is clear in this example that the LMI optimization allows to find a much tihgter upper bound.

5 Discussion and concluding remarks

It must be noted that in the above example the performance indices have the same control weighting matrix R. Thus, by taking the same value of the scalar δ one obtains one value of feedback gain. This is not a general case. Different values of R_i, δ_i will give different values of feedback gains for the same P and this seems to deviate from the initial problem statement. In effect, for any choice of δ_i, R_i, $i = 1, ..., r$ the control law (15) guarantees the upper bound (16) for all performance objectives. Moreover, by chosing the suitable feedback gain between them it is possible to satisfy further design specifications.

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